Intertemporal Consumption and Portfolio Investment

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Abstract

This paper derives analytic solutions to the optimization problem of a consumer who satisfies the Kreps-Proteus preferences. To achieve this, the aggregator function of Epstein and Zin is adopted and the certainty equivalent of future utility is defined to be that of the negative exponential function. In addition, it is assumed that in any period the certainty equivalent of future utility depends on initial wealth. These modifications lead to a sequence of two-period optimization problems. In each period the consumer determines current consumption and the certainty equivalent of next period's wealth. The problem of the consumer becomes more transparent and a possible explanation to the equity premium puzzle is offered.

Key words: Recursive utility, non-expected utility theory, portfolio theory, risk aversion, computational economics.

1 Preamble

This paper studies optimal consumption-portfolio decisions of an economic agent. It deals with a consumer who satisfies the Kreps-Proteus preferences, and it derives analytic solutions to a special case of this problem. To achieve this, the aggregator function of Epstein and Zin relating to these preferences is adopted , and the certainty equivalent of future utility is defined to be that of the negative exponential function. In addition, is it assumed that in any period the certainty equivalent of future utility depends on the wealth of the consumer at the beginning of that period; thus, the attitude to risk varies over time. These modifications lead to a recursive optimization problem which has a closed form solution. It is shown that the recursive problem is essentially a sequence of two-period optimization problems where, in each period, the agent determines current consumption and the certainty equivalent of the agent's wealth for the next period. The results render the problem of the consumer more transparent and offer a possible explanation to the equity premium puzzle based on a time-varying attitude to risk.

1.1 Temporal Preferences

Selden (1978), (1979) studied the problem of optimal consumption-saving decisions in a two period model. He assumed that the objects of choice of the agent are present consumption and the set of cumulative distribution functions of future consumption. Preferences over future consumption are representable by a Neumann-Morgestern (NM) index, and may depend on the value of some variable which is known with certainty, such as the level of present consumption (called "risk preference dependence"). Selden shows that, in this setting, consumption preferences can be represented by an ordinal utility function having as arguments present consumption and the certainty equivalent of future consumption bundles. However, his axioms do not guarantee that the choices of the consumer are time-consistent; in a multi-period setting, optimal policies adopted in the first period will be revised in the future.

The inconsistency problem just mentioned was rectified by Kreps and Proteus (KP) (1978) who assumed that the objects of choice of the consumer are intertemporal lotteries. These lotteries are essentially cumulative distribution functions, conditional on the consumption history the agent up to period t; they represent the probability of receiving at t+1 an uncertain amount of consumption and a ticket to another lottery which takes place at t+2. The preferences over such lotteries, satisfy the NM axioms with the exception of a special case of the independence axiom: compound lotteries cannot be reduced to simple ones if such reduction alters the timing at which information is revealed. The reason for this is the possibility that agents may not be indifferent between early and late resolution of the uncertainty concerning their final consumption. In this framework KP showed that the choices of the consumer may be represented by a sequence of utility functions each of which has two arguments: the consumption at period t and the maximum expected utility that is attainable in the future.

Epstein and Zin (EZ) (1989) generalized the KP approach by introducing temporal lotteries which can be thought-of as infinite probability trees in which each node corresponds to the certainty equivalent of the maximum utility which can be attained in the future. Certainty equivalents are evaluated by means of a broad class of mean value functionals. Preferences between present consumption and the certainty equivalent of future utility are represented by a function (called the "aggregator function") which has a particular functional form. The EZ approach, by contrast to the usual expected utility approach, separates the measure of risk aversion from the elasticity of intertemporal substitution, and has testable implications concerning several key issues in this area (see also Epstein and Zin 1990, 1991, Kocherlakota,1996). Farmer (1990) exploits the EZ parametrization in a finite horizon model with one risky financial asset and a random endowment of income in every period. He derives closed-form solutions to the problem of the consumer in a case which corresponds to a form of risk neutrality.

In this paper, it is assumed that the preferences of the consumer satisfy KP axioms and the EZ functional form of the aggregator function is adopted. Unlike Epstein and

Zin, certainty equivalents are evaluated by means the negative exponential function. Further, it is assumed that in any period the certainty equivalent of future utility depends on the wealth which the consumer has at the beginning of that period. This makes the attitude to risk time dependent, as will be explained below. It is shown (by construction) that in this framework choices are time-consistent.

1.2 The Endowment Effect

In the early literature it was suggested that the ordering of atemporal prospects should be conditional on the initial amount of wealth which the individual owns. Experiments conducted by Davidson, Suppes and Siegel (1957) suggest that risk preferences about different wealth levels are indicative of a shifting utility function, rather than movements along a fixed utility function. Similarly, the evidence provided by Binswanger (1981) supports the hypothesis of a shifting utility function "anchored" to the level of initial wealth.

Thaller (1980) conducted experiments which showed that people often demand a much higher price to give up an object than they would pay to acquire it. He calls this pattern the "endowment effect". Samuelson and Zeckhauser (1988) refer to such behaviour as the "status quo bias", and Kahneman and Tversky (1984) as the "loss aversion " attitude. This anomaly in preferences has been differently interpreted by the authors mentioned above. It could also be interpreted as a shift in the utility function caused by the acquisition of the object in question.

The discussion thus far refers mainly to atemporal preferences. Selden (1978) allowed for "risk preference dependence" into his two-period model : His axioms allow preferences over second-period distribution functions to depend on the value of some variable which is known with certainty in the first period. In a multi-period model this approach would imply that risk preferences may be anchored on a variable which changes at the beginning of every period.

Here it is assumed that the "anchoring" point of the NM utility functional in any period is the initial wealth that the individual owns at the beginning of that period. True, the individual knows that initial wealth will change in the future. However, the direction of change is not certain. Further, if one accepts Koopmans' (1962) flexibility of future preferences (with respect to baskets of consumption commodities), the idea becomes attractive: need for flexibility may arise because preferences about certain subsets of the commodity space may be "fuzzy". The initial wealth of a period may eliminate some of this fuzzines by allowing the consumer to explore those subsets which can be afforded. In this case, the induced preferences over the utility of future consumption plans would depend on wealth.

The order of discussion is as follows: Section 2 develops the representation of the preferences of a consumer, and describes the environment in which the consumer operates. Section 3 contains the main results of the paper, and their interpretation (section 3.1). In order to exploit the logic of the solution to the problem of the

consumer, a series of simulations have been performed. These are also presented end discussed in Section 3. Proofs of the main results are presented in Appendix A so as not to interrupt the flow of the argument. Similarly, details of the simulations are given in appendix B.

2 Preferences and the Environment

2.1 Representation of Preferences

Let $(C_0, C_1, ..., C_{T+1})$ be a deterministic consumption sequence. In order for preferences over such sequences to be time-consistent the utility (V) has to be recursive; i.e. V must satisfy the relation:

$$V_0(C_0, C_1, ..., C_{T+1}) = u(C_0, V_1(C_1, C_2, ..., C_{T+1}),$$

where u is the aggregator function, and V_1 has the same functional form as V_0 (Koopmans, 1960).

When dealing with random consumption streams, the utility which can be attained from t+1 onward is random. Following Epstein and Zin, it is assumed that the consumer evaluates the certainty equivalent of future utility, $\mu_t[\widetilde{V}_{t+1})]$, and then combines it with current consumption via the aggregator function u. This gives lifetime utility:

$$\widehat{V}_t = u(C_t, \mu_t[\widetilde{V}_{t+1}]),$$

where $\widetilde{()}$ denotes a random variable and $\widehat{()}$ the certainty equivalent of a random variable. Similarly (Epstein and Zin, 1989), u is specified to be

$$u(C, z) = (C^\rho + \beta z^\rho)^{\frac{1}{\rho}},$$

$0 < \beta < 1, \rho \neq 0$, $\rho < 1$. However, unlike Epstein and Zin, the certainty equivalent of a random variable $\widetilde{x}_{t+1}$ is specified to be:

$$\mu[\widetilde{x}_{t+1}] \equiv -\frac{W_t}{\lambda} \ln E[(\exp(-\frac{\lambda}{W_t}\widetilde{x}_{t+1}))],$$

for some constant λ. Combining the previous expressions there results:

$$\begin{aligned}\widehat{V}_t &= (c_t^\rho + \beta\mu[\widetilde{V}_{t+1}]^\rho)^{\frac{1}{\rho}} \\ &= (C_t^\rho + \beta(-\frac{W_t}{\lambda}\ln E[(\exp(-\frac{\lambda}{W_t}\widetilde{V}_{t+1}))])^\rho)^{\frac{1}{\rho}}.\end{aligned}$$

Here, β is the time discount factor and $1/(1-\rho)$ is the elasticity of intertemporal substitution.

It will be shown (by construction) that the functional form of V_{t+1} the same as that of V_t; thus, preferences are time-consistent.

2.2 Representation of the The Environment

Consider the economic environment of a consumer with time horizon t = 0,1,...,T+1. At the beginning of each period the consumer owns a certain amount of financial assets which consist of a risk-free bond and risky shares to a firm. The consumer also receives labour income; current labour income is known with certainty, future income is random.

The initial wealth at time t consists of returns from assets and the current labour income (A_t), and the certainty equivalent of future labour income (H_t). Initial wealth is either consumed (C_t), or invested in shares (S_t) and bonds (B_t) :

$$\begin{aligned} W_t &= A_t + H_t. \\ &= C_t + B_t + S_t. \end{aligned}$$

Similarly, in period t+1:

$$\begin{aligned} \widetilde{W}_{t+1} &= B_t r + S_t \widetilde{r}_t + \widetilde{Y}_{t+1} + \widetilde{H}_{t+1} \\ &\equiv \widetilde{A}_{t+1} + \widetilde{H}_{t+1}, \end{aligned}$$

where $\widetilde{r}_t$, and r are the returns of shares and bonds, respectively. Short sales of assets are permitted. Eliminating B_t from the previous expressions results in:

$$\widetilde{A}_{t+1} = (A_t - C_t)r + (\widetilde{r}_t - r)S_t + \widetilde{Y}_{t+1}.$$

The certainty equivalent of $\widetilde{H}_{t+1}$ cannot be readily evaluated under the conditions of the model An assumption is introduced which is motivated by the fact that in practice one cannot borrow against labour income of the distant future. It is assumed that income future to one period ahead is taxed away; i.e. in period t+1 a tax (which is random in t) is imposed which is equal to $\widetilde{H}_{t+1}$, $t = 0, 1, ..., T$. Thus,

$$W_t = A_t + H_t, \text{ and } \widetilde{W}_{t+1} = \widetilde{A}_{t+1},$$

thus, H_t consists of the present vale of the certainty equivalent of the labour income of period $t + 1$ only.

3 Dynamic Optimization

Consider period $T + 1$, at which time we suppose all uncertainty has been resolved. Define:

$$V_{T+1} \equiv C_{T+1} = W_{T+1} = A_{T+1}.$$

Starting from period T, a sequence of functions $\{V_t\}_{t=0}^{T}$ is constructed recursively :

$$\widehat{V}_t = \max_{\{C_t, S_t\}} (C_t^\rho + \beta(-\frac{W_t}{\lambda} \ln E[(\exp(-\frac{\lambda}{W_t}\widetilde{V}_{t+1}))])^\rho)^{\frac{1}{\rho}}$$

subject to

$$\widetilde{A}_{t+1} = (A_t - C_t)r_t + (\widetilde{r}_t - r_t)S_t + \widetilde{Y}_{t+1},$$

$t = T, T-1, ..., 0$, where $W_t = A_t + H_t$.

To pave the way to the main theorem, the following lemma is required:

Lemma 1 *In period t the certainty equivalent of a portfolio of shares S_t and the labour income of period t+1 ($\widetilde{Y}_{t+1}$), $t = 0, 1, ..., T$ are, respectively,*

$$-\frac{W_t}{\lambda} \ln(E[\exp(-\frac{\lambda}{W_t} g_{t+1} S_t \widetilde{r}_t)]) = S_t g_{t+1} \widehat{R}_t$$

$$-\frac{W_t}{\lambda} \ln(E[\exp(-\frac{\lambda}{W_t} g_{t+1} \widetilde{Y}_{t+1})]) \equiv g_{t+1} \widehat{Q}_{t+1},$$

where,

$$g_t = f_t h_t$$

$$c_t \equiv \phi_t h_t$$

$$h_t = r + s_t(\widehat{R}_t - r)$$

$$\theta_t \equiv \frac{\lambda}{W_t} g_{t+1} S_t$$

$$s_t \equiv \frac{S_t}{W_t} = \frac{\theta_t}{\lambda g_{t+1}}$$

$$\widehat{R}_t \equiv \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i!} (\theta_t)^{i-1} k_{ti} - r))$$

$$\widehat{Q}_{t+1} \equiv \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i!} (\frac{\lambda}{W_t} g_{t+1})^{i-1} l_{t+1i}$$

$$f_t = [\phi_t^\rho + \beta g_{t+1}^\rho (1 - \phi_t r)^\rho]^{\frac{1}{\rho}}$$

$$\phi_t = \frac{(\beta r g_{t+1}^\rho)^{\frac{1}{\rho-1}}}{1 + r(\beta r g_{t+1}^\rho)^{\frac{1}{\rho-1}}}$$

$$g_{T+1} = f_{T+1} = h_{T+1} = 1,$$

and where k_{ti} and l_{t+1i} are the ith cumulalnts of the conditional distribution function of $\widetilde{r}_{t+1}$ and. $\widetilde{Y}_{t+1}$, respectively

Proof. See Appendix A. In Lemma 1, h_t is (one plus) the rate of return on the agent's optimum portfolio. $\widehat{R}_t$ is the certainty equivalent of a dollar invested in shares, and s_t is the proportion of the individual's funds invested in shares. The expressions ϕ_t and f_t are factors by which future consumption and utility, respectively, are "discounted" to the present. It should be noted that $(c_t, s_t, g, h_t, f_t))$, are non-stochastic and they are derived from the cumulants of the conditional distributions of $\widetilde{r}_t, \widetilde{r}_{t+1}, ..., \widetilde{r}_T$.

Theorem 1 *In period t, t = 0, 1,.., T, exist constants* $(c_t^*, \theta_t^*, g_{t+1}^*)$ *such that:*

$$C_t = c_t^* W_t$$

$$S_t = \frac{W_t}{\lambda g_{t+1}^*}\theta_t^*,$$

$$\widehat{V}_t = g_t^* W_t,$$

where

$$\theta_t^* = \max(\text{real } \ root(\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{(i-1)!}{\theta_t}^{i-1} k_{ti} - r)),$$

and (c_t^*, g_{t+1}^*) *are as defined in Lemma I with* θ_t *is replaced by* θ_t^*. *Thus, at the optimum:*

$$s_t \equiv \frac{S_t}{W_t} = \frac{\theta_t^*}{\lambda g_{t+1}^*} = constant.$$

Proof. See Appendix A. It follows from Theorem 1 that consumption, investment in risky assets and the maximum value function are all linear functions of wealth. The propensity to consume and the propensity to invest in shares are each non-random; randomness in this model stems from the sources and not the disposition of wealth. In order to simlpify the notation the superscript (*) will be omitted from the remaining of the paper; the understanding is that variables such as $g, f, h, \phi, c, \widehat{R}, s$, are evaluated at θ^*, unless otherwise stated.

3.1 Implications

From the proof of Theorem 1 it follows that the recursive problem of this section is essentially a sequence of two-period optimization problems: In each period the agent determines current consumption and the certainty equivalent of the agent's wealth for the next period. Thus, what holds true for only two periods in Selden's model, here holds in a multi-period setting. The certainty equivalent of future wealth is determined by means of the negative exponential *utility* function, where

$$\Lambda_t \equiv \frac{\lambda}{W_t} g_{t+1}$$

is the one-period measure of risk aversion. For the purpose of clarity Λ_t will be refer to as the "period measure of risk aversion", or the "p-measure" in short. It varies with time and it depends on environment of the individual, i.e. on expectations about future rates of return and remaining life-time (through g_{t+1}), as well as the agent's past history of consumption (through W_{t+1}). The parameter λ can be interpreted as a component of the risk preferences which is independent of the environment. λ will be refer to as the "life-time measure of risk aversion", or the "l-measure" in short.

Consider a group of individuals who trade financial assets under the conditions of this model, and let superscripts denote a particular individual.

Lemma 2 *Assume that for any two individuals, the jth and the kth, the conditional distributions of $\tilde{r}_t$, $t = 0, 1, 2, ..., T,$ are equal; therefore $k^j_{it} = k^k_{it}$ for all t. Then, in a competitive equilibrium the value of θ_t :*

$$\theta_t = \Lambda_t S_t \tag{1}$$

is the same for all market participants.

Proof. See Appendix A.

An immediate consequence of Lemma 2 is the following. Let $S^m_t \equiv \Sigma_j S^j_t$ be the total supply of shares in period t.

Corollary 1 *The amount invested in shares by the kth individual is:*

$$S^k_t = \frac{S^m_t}{\frac{\lambda^k g^k_{t+1}}{W^k_t} \sum_j \frac{W^j_t}{\lambda^j g^j_{t+1}}} \equiv \frac{S^m_t}{\Lambda^k_t \sum_j \frac{1}{\Lambda^j_t}}.$$

Thus, the "p-measure of risk aversion" of an individual relative to that of all other members of the community determines the amount of shares the individual holds in equilibrium.

From Corollary 1 it follows immediately that

Corollary 2

$$r = \Sigma_i \pi_{ti} k_{ti}$$

where,

$$\pi_{it} \equiv \frac{(-1)^i}{(i-1)!} \theta^{i-1}_t k_{ti}.$$

In Corollary 2 π_{it} is the contribution of the i th cumulant to the interest rate of the period, and it can be considered as the price which the market attaches to that cumulant. It is known for some time that in atemporal choices and with a constant measure of absolute risk aversion, the equilibrium price of an asset has a special structure: it is a linear combination of the cumulants of the distribution of returns. Thus, cumulants enter as separate commodities (see Borch, 1962). This property is also valid in temporal choices.

3.1.1 Simulations.

In Appendix B the details of a simulation model are discussed, and the graphs of several runs are presented.in Appendix B. Here some observations which follow are summarized.

The variation of parameters β, ρ, and λ has minor effects on the marginal propensity to consume; for other variables it has, in general, the anticipated effects: Increasing the time discount factor β, makes the consumer to want to save more in earlier years. Similarly, increasing ρ, increases the elasticity of intertemporal substitution and causes a more uneven pattern of consumption in favour of consumption for latter years. To achieve this, the consumer takes greater risks early on by investing more in shares, and less in bonds (as $\rho \to 1$, the consumer invests exclusively in shares). On the other hand, decreasing ρ leads to a more even pattern of consumption which is achieved by investing more in the secure assets, i.e. bonds (as $\rho \to 0$ the consumer invests almost exclusively in bonds).

Different values of (β, λ, ρ) yield simulated results which are more or less reasonable depending on the composition of initial wealth: Suppose that the consumer relies initially mostly on assets to generate present and future consumption. Then, the results are more reasonable the higher is the discount factor β, i.e. the more patient the consumer is (e.g. $\beta = .99$). On the other hand, consumers who rely more on labour income exhibit reasonable simulated behaviour if they are more impatient (e.g. $\beta = .9$).

The parameters λ and ρ play similar roles in the following sense: Suppose we are given (β, λ, ρ) to compute the optimal paths of $(C_t,\ S_t.)$, and λ is allowed to increase to λ'. For any λ' a ρ', $\rho' < \rho$, can be founf such that the paths computed with (β, λ', ρ) and (β, λ, ρ'), respectively, remain close to each other for most of the time. The implication is that the life-time attitude to risk cannot be disentangled entirely from the elasticity of intertemporal substitution. The inseparability of the two concepts was first noted by Epstein-Zin (1987).

Consider the one-period measure of risk aversion Λ_t. The greater is Λ_t the less one invests in shares in any given period. Further, the p-measure varies directly with λ and inversely with ρ. The effect of β differs for earlier years than for later ones: The greater is β, the greater is the Λ_t in earlier years than in later ones. As mentioned above, higher β makes one to want to save more in earlier years; in those years the consumer also appears to be more risk averse.

The result just mentioned may provide a plausible explanation for the equity premium puzzle in the following sense: In the simulations the l-measure and the p-measure have reasonable numerical values when ρ lies in the interval $[0.4, 0.8]$ and β in $[0.9, 0.99]$; the model does not require unreasonably high l-measures to risk in order to yield investment in bonds and shares. In addition, the time profile of the p-measure can be either increasing or decreasing. Therefore, highly l-risk averse individuals may appear to be more or less so as they approach the end of their time horizon. Consequently, an overlapping generations model would imply that the

equity-premium puzzle is a consequence of the age composition of the population. (The model developed here cannot be readily extended to an infinite horizon one. However, it can be extended to an overlapping generations model).

Appendix A

Proof of Lemma 1 and Theorem

Suppose that in period t, $t = 0, 1, ..., T$, the consumer believes that in period T+1 all uncertainty with respect to W_{T+1}will be resolved. Therefore,

$$V_{T+1} \equiv C_{T+1} = W_{T+1} = A_{t+1},$$

Knowing this, his or her maximum value function for period T will be:

$$\begin{aligned}
\widehat{V}_T &= \max_{\{C_T, S_T\}} \{C_T^\rho + \mu[\widetilde{V}_{T+1}]^\rho\}^{\frac{1}{\rho}} \\
&= \max_{\{c_T, S_T\}} \{C_T^\rho + \beta[-\frac{W_T}{\lambda}\ln(E[\exp(-\frac{\lambda}{W_T}\widetilde{W}_{T+1})])]^\rho\} \quad ,
\end{aligned}$$

where,

$$\widetilde{W}_{T+1} = \widetilde{A}_{T+1} = (A_T - C_T)r + (\widetilde{r}_T - r)S_T + \widetilde{Y}_{T+1}.$$

Substitute for $\widetilde{W}_{T+1}$ in $\widehat{V}_T$:

$$\begin{aligned}
&= \max_{\{c_T, S_T\}} \{C_T^\rho + \beta[-\frac{W_T}{\lambda}\ln(E[\exp(-\frac{\lambda}{W_T}((A_T - C_T)r \\
&\quad +(\widetilde{r}_T - r)S_T + \widetilde{Y}_{T+1}))])]^\rho\},
\end{aligned}$$

and consider the expression in the square brackets (after expanding for the terms under the logarithm):

$$\begin{aligned}
&= -\frac{W_T}{\lambda}\ln(\exp(-\frac{\lambda}{W_T}((A_T - C_T)r) * E[\exp(-\frac{\lambda}{W_T}(\widetilde{r}_T - r)S_T)] \\
&\quad *E[\exp(-\frac{\lambda}{W_T}\widetilde{Y}_{T+1})]) = (A_T - C_T)r - \frac{W_T}{\lambda} \\
&\quad * \ln(E[\exp(-\frac{\lambda}{W_T}(\widetilde{r}_T - r)S_T)] * E[\exp(-\frac{\lambda}{W_T}\widetilde{Y}_{T+1})]).
\end{aligned}$$

The second equality follows from the assumption of independence between $\widetilde{r}_T$ and $\widetilde{Y}_{T+1}$.

To proceed, assume that each of $\widetilde{r}_T$ and $\widetilde{Y}_{T+1}$ follow some stochastic processes which the consumer has identified and uses to determine the cumulants of their conditional distributions from period t to $T+1$. Then, follow Kendal and Stuart (1958)

and Cramer (1963): Suppose that. $\frac{\lambda}{W_t}$ is a small number. Let k_{Ti}, and l_{Ti} be the cumulants of the conditional distribution of $\tilde{r}_T$ and $\tilde{Y}_{T+1}$, respectively, and consider the RHS of the last expression. Expand the terms under the expectation signs in Maclaurin series around zero. Finally, expand the logarithm. The end result is:,

$$\begin{aligned} &= S_T(\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i!}(\frac{\lambda}{W_T}S_T)^{i-1}k_{Ti} - r) + \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i!}(\frac{\lambda}{W_T})^{i-1}l_{T+1i} \\ &\equiv S_T(\widehat{R}_T - r) + \widehat{Q}_{T+1}, \end{aligned}$$

where,

$$\widehat{R}_T \equiv \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i!}(\frac{\lambda}{W_T}S_T)^{i-1}k_{Ti},$$

and

$$\widehat{Q}_{T+1} \equiv \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i!}(\frac{\lambda}{W_T})^{i-1}l_{T+1i}.$$

Here $\widehat{Q}_{T+1}$ is the certainty equivalent of the labour income of period T+1 evaluated at the end of period T. Finally, substitute these expressions back in the maximum value function, to derive:

$$\widehat{V}_T = \max_{\{C_T, S_T\}} \{C_T^{\rho} + \beta[(A_T - C_T)r + S_T(\widehat{R}_T - r) + \widehat{Q}_{T+1}]^{\rho}\}^{\frac{1}{\rho}}.$$

The optimality conditions of this expression with respect to S_T and C_T are, respectively,

$$\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{(i-1)!}(\frac{\lambda}{W_T}S_T)^{i-1}k_{Ti} = r,$$

and,

$$(1 + r(\beta r)^{\frac{1}{\rho-1}})C_T = (A_T r + S_T(\widehat{R}_T - r) + \widehat{Q}_{T+1})(\beta r)^{\frac{1}{\rho-1}}.$$

Next, define:

$$\frac{\lambda}{W_T}S_T \equiv \theta_T, \text{ and } H_T \equiv \frac{1}{r}\widehat{Q}_{T+1},$$

and substitute for the corresponding expressions in the two optimality conditions. Solve first condition for θ_t:

$$\theta_T^* = \max(\text{real root}(\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i!}{\theta_T}^{i-1}k_{Ti} - r)).$$

Use this result in the second condition and solve it for C_T :

$$\begin{aligned} C_T &= \frac{(\beta r)^{\frac{1}{\rho-1}}}{1 + r(\beta r)^{\frac{1}{\rho-1}}}[A_T r + (\widehat{R}_t - r)S_T + rH_T] \\ &\equiv \phi_T[A_T r + (\widehat{R}_t - r)S_T + rH_T], \end{aligned}$$

and since $W_T = A_T + H_T$, the above expression becomes:

$$\begin{aligned} &= \phi_T[(A_T + H_T)r + (\widehat{R}_t - r)s_T W_T] \\ &= \phi_T W_T [r + (\widehat{R}_t - r)s_T] \\ &= \phi_T h_T W_T \equiv c_T W_T. \end{aligned}$$

Finally, substitute the above expressions in the maximum value function:

$$\begin{aligned} \widehat{V}_T &= [\phi_T^\rho + \beta(1-\phi_T r)^\rho]^{\frac{1}{\rho}} h_T W_T \\ &\equiv f_T h_T W_T \equiv g_T W_T. \end{aligned}$$

In period T-1 the value function becomes:

$$\begin{aligned} \widehat{V}_{T-1} &= \max_{\{C_{T-1}, S_{T-1}\}} \{C_{T-1}^\rho + \beta(\widetilde{V}_T)^\rho\}^{\frac{1}{\rho}} \\ &= \max_{\{c_{T-1}, S_{T-1}\}} \{C_{T-1}^\rho + \beta[-\frac{W_{T-1}}{\lambda}\ln(E[\exp(-\frac{\lambda g_T}{W_{T-1}}\widetilde{W}_T)])]^\rho\}^{\frac{1}{\rho}}, \end{aligned}$$

where

$$\widetilde{W}_T = \widetilde{A}_T = (A_{T-1} - C_{T-1})r + (\widetilde{r}_{T-1} - r)S_{T-1} + \widetilde{Y}_T.$$

After replacing $\widetilde{W}_T$ with its equal the maximum value function becomes:

$$\begin{aligned} = &\max_{\{c_{T-1}, S_{T-1}\}} \{C_{T-1}^\rho + \beta[-\frac{W_{T-1}}{\lambda}\ln(E[\exp(-\frac{\lambda g_T}{W_{T-1}}((A_{T-1} - C_{T-1})r \\ &+(\widetilde{r}_{T-1} - r)S_{T-1} + \widetilde{Y}_T)])]^\rho\}^{\frac{1}{\rho}}. \end{aligned}$$

Proceeding as in period T, the expression under the first set of square brackets is:

$$\begin{aligned} = &-\frac{W_{T-1}}{\lambda}\ln\{\exp(-\frac{\lambda g_T}{W_{T-1}}((A_{T-1} - C_{T-1})r) * E[\exp(-\frac{\lambda g_T}{W_{T-1}}(\widetilde{r}_{T-1} - r)S_{T-1})] \\ &*E[\exp(-\frac{\lambda g_T}{W_{T-1}}\widetilde{Y}_T)]\} \\ = &\ g_T(A_{T-1} - C_{T-1})r + \sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{i!}(\frac{\lambda}{W_{T-1}})^{i-1}(S_{T-1}g_T)^i k_{T-1i} - r g_T S_{T-1} \\ &+\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{i!}(\frac{\lambda}{W_{T-1}})^{i-1} g_T^i l_{Ti} \\ = &\ g_T[(A_{T-1} - C_{T-1})r + S_{T-1}(\widehat{R}_{T-1} - r) + \widehat{Q}_T], \end{aligned}$$

where

$$\widehat{R}_{T-1} \equiv \sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{i!}(\frac{\lambda g_T}{W_{T-1}}S_{T-1})^{i-1}k_{T-1i},$$

and

$$\widehat{Q}_T \equiv \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i!} (\frac{\lambda g_T}{W_{T-1}})^{i-1} l_{T-1i}.$$

By using these expressions the maximum value function becomes:

$$\begin{aligned} \widetilde{V}_{T-1} &= \max_{\{C_{T-1}, S_{T-1}\}} \{C_{T-1}^{\rho} + \beta g_T^{\rho}[(A_{T-1} - C_{T-1})r + S_{T-1}(\widehat{R}_{T-1} - r) \\ &+ \widehat{Q}_T]^{\rho}\}^{\frac{1}{\rho}}, \end{aligned}$$

The optimality conditions with respect to $S_{T-1}$and C_{T-1} are:

$$\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{(i-1)!} (\frac{\lambda g_T}{W_{T-1}} S_{T-1})^{i-1} k_{T-1i} = r,$$

and,

$$(1 + r(\beta r)^{\frac{1}{\rho-1}})C_{T-1} = (A_{T-1}r + S_{T-1}(\widehat{R}_{T-1} - r) + \widehat{Q}_T)(\beta r)^{\frac{1}{\rho-1}},$$

respectively. To proceed, define, as before,

$$\frac{\lambda}{W_{T-1}} S_{T-1} = \theta_{T-1},$$

and substitute for the corresponding expressions in the two first order conditions. Solve the first condition for θ_{T-1}:

$$\theta_{T-1}^* = \max(\text{real root}(\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i!} \theta_{T-1}^{i-1} k_{T-1i} - r)).$$

Thus, at the optimum:

$$s_{T-1} \equiv \frac{S_{T-1}}{W_{T-1}} = \frac{\theta_{T-1}^*}{\lambda g_T} = constant,$$

and

$$\frac{\lambda g_T}{W_{T-1}} S_{T-1} = \theta_{T-1}^*.$$

Define, as before, $H_{T-1} \equiv \frac{1}{r}\widehat{Q}_T$ and solve the second condition for C_{T-1} :

$$\begin{aligned} C_{T-1} &= \frac{(\beta g_T^{\rho} r)^{\frac{1}{\rho-1}}}{1 + r(\beta g_T^{\rho} r)^{\frac{1}{\rho-1}}} [A_{T-1} r_{T-1} + (\widehat{R}_{t-1} - r)S_{T-1} + rH_{T-1}] \\ &\equiv \phi_{T-1}[(A_{T-1} + H_{T-1})r + (\widehat{R}_{t-1} - r)S_{T-1}] \\ &= \phi_{T-1} W_{T-1}[r + (\widehat{R}_{t-1} - r)s_{T-1}] \\ &\equiv \phi_{T-1} h_{T-1} W_{T-1} \equiv c_{T-1} W_{T-1}. \end{aligned}$$

Finally, upon substitution in the value function we obtain:

$$\begin{aligned}\widehat{V}_{T-1} &= [\phi_{T-1}^{\rho} + \beta g_T^{\rho}(1-\phi_{T-1}r)^{\rho}]^{\frac{1}{\rho}} h_{T-1}W_{T-1} \\ &\equiv f_{T-1}h_{T-1}W_{T-1} \equiv g_{T-1}W_{T-1},\end{aligned}$$

since $W_{T-1} = A_{T-1} + H_{T-1}$

Proceeding in a similar manner for periods $T-2, T-1, ..., t$ (where $t = 0, 1, ..., T$) the conditions of Theorem 1 and Lemma 1 are derived.

Proof of Lemma 2

In order to prove the Lemma it must be shown that the expression:

$$\ln(E[\exp(-\theta_t \widetilde{r}_t]) = \sum_{i=1}^{\infty} \frac{(-1)^i}{i!}\theta_t^i k_{ti},$$

is monotonic in the neighborhood of S_t. Given any two amounts θ_t^1 and θ_t^2 the following relation holds:

$$\begin{aligned}& E[\exp(-\{\theta_t^1 + \theta_t^2\}\widetilde{r}_t) \\ \leq\ & \frac{1}{2}E[\exp(-\theta_t^1\widetilde{r}_t)] + \frac{1}{2}E[\exp(-\theta_t^2\widetilde{r}_t)],\end{aligned}$$

by Hölder's inequality. By taking the logarithm of both sides one obtains.

$$\begin{aligned}& \ln(E[\exp(-\frac{\lambda}{W_t}g_{t+1}\{\theta_t^1 + \theta_t^2\}\widetilde{r}_t)) \\ \leq\ & \frac{1}{2}\ln((E[\exp(-\theta_t^1\widetilde{r}_t)])) + \frac{1}{2}\ln((E[\exp(-\theta_t^2\widetilde{r}_t)])),\end{aligned}$$

which proves convexity. Furthermore, the first order conditions of maximization of:

$$\sum_{i=1}^{\infty} \frac{(-1)^i}{i!}\theta_t^i k_{ti} = S_t r$$

with respect to θ_t are:

$$\frac{d}{dS_t}\ln(E[\exp(-\theta_t\widetilde{r}_t)]) = \sum_{i=1}^{\infty} \frac{(-1)^i}{(i-1)!}{\theta_t}^{i-1}k_{ti} = r,$$

therefore,

$$\sum_{i=1}^{\infty} \frac{(-1)^i}{i!}\theta_t^{*i}k_{ti}$$

is monotonically increasing in the neighborhood of the equilibrium θ_t^* . Thus, for two individuals, the ith and the kth we have

$$\theta_t^{*i} \equiv (\frac{1}{W_t}\lambda S_t g_{t+1})^i = (\frac{1}{W_t}\lambda S_t g_{t+1})^k \equiv \theta_t^{*k}$$

This proves Lemma 2. A little manipulation gives the results of Corollary 1.

Proof of Corollary 2

The proof follows immediately from the first order maximization conditions with respect to θ_t:

$$\sum_{i=1}^{\infty} \frac{(-1)^k}{(i-1)!}\theta_t^{i-1}k_{ti} = r,$$

by defining

$$\pi_{ti} = \frac{(-1)^k}{(i-1)!}\theta_t^{ki-1}k_{ti}.$$

Appendix B

The statistics for the simulations have been taken from Kocherlakota (1996) they refer to US annual data for 1889-1978. They are as follows:

$$r = 0.01,\ \text{mean}(\widetilde{r}) = 0.070,\ \text{Variance}(\widetilde{r}) = 0.0274$$

The variance of r and the covariance between r and $\widetilde{r}$ reported in Kocherlakota have been ignored. We have not attempted to estimate higher moments and the study their effect on the optimal policies, even though the model allows this possibility.

The time-series process which generated these observations is assumed to be an AR(1) process:

$$\begin{aligned} y_t - \mu &= a(y_{t-1} - \mu) + \varepsilon_t, \\ \text{where } \mu &= 0.07,\quad a = 0.6,\quad \varepsilon_t \sim N(0, \sigma^2), \text{.and } \sigma^2 = \frac{1}{1-\phi^2} var(y_t) \end{aligned}$$

It should be noted that the marginal propensities (c_t, s_t) of a given period t are non-random; they are derived by iterating (g, h, f) backward from period to T+1 to t. This requires that the cumulants of the conditional distributions of $\widetilde{r}_{T+1}, \widetilde{r}_T, ..., \widetilde{r}_t$ are known. It is assumed that the consumer has identified the AR process mentioned above and uses this model to determine the moments of the conditional distributions of $\widetilde{r}$ from period t to $T+1$ Then, by iterating backwards he or she arrives at (c_t, s_t).

References

[1] Binswanger, H. (1981), "Attitudes Towards Risk: Theoretical Implications of an Experiment in Rural India, " *Economic Journal*, 91, pp.867-890.

[2] Borch, K. (1952), "Programming Under Uncertainty and the Price of Risk", *Zeitschrift fur Nationalekonomie, pp.278-291.*

[3] Cramer, H. (1963), *Mathematical Methods of Statistics,* Princeton University Press, pp.89-103

[4] Davidson Suppes and Siegel, S. (1957) *Decision Making: An Experimental Approach*, Stanford: Stanford University Press.

[5] Epstein, L. and Zin S., (1989), "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns I: A Theoretical Framework," *Economertica*, 53, pp. 937-969.

_________, "First order Risk Aversion and the Equity Premium Puzzle," *Journal of monetary Economics, 26,* pp.387-407.

_________,(1991) "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns II:An Empirical Analysis," *Journal of Polotical Economy, 92*, pp.263-286.

[6] Farmer, R. E., (1990) "RINCE Preferences," *Quarterly Journal of Economics*, 420, pp.43-60.

[7] Kahneman, D. and Tversky, A. (1979) "Prospect Theory: An Analysis of Decisions under Risk, " *Econometrica*, 47, pp. 263-291.

[8] Kendal, M. and Stuart, A.,(1958) *The Advanced Theory of Statistics, Vol.1,* Hafner, pp.67-84.

[9] Kocherlakota, N. R., (1996) "The Equity Premium: It is Still a Puzzle," *Journal of Economic Literature*, XXXIV, pp.42-71

[10] Koopmans, T. C., (1960), "Stationary Ordinal Utility and Impatience, " *Econometrica*, 28, pp. 278-309.

__________ , (1962) "On Flexibility of Future Preferences," *Cowles Foundation,* Discussion Paper No 150.

[11] Kreps, D. M., and Proteus, E. M., (1978), "Temporal Resolution of Uncertainty and Dynamic Choice Theory, " *Economertica*, 46, pp.185-200.

___________, (1979a), "Temporal Temporal von Neumann-Morgestern and Induced Preferences," *Journal of Economic Theory*, 20, pp.139-171.

___________, (1979b), "Dynamic Choice Theory and Dynamic Programming, " *Econometrica*, 47, pp.91-100

[12] Samuelson, W. and Zeckhauser, R. (1988), "Status Quo Bias in Decision Making, " *Journal of Risk and Uncertainty*, 1, pp. 7-59.

[13] Selden, L. (1978), "A New Representation of Preferences over 'Certain $\times$ Uncertain' Consumption Pairs: The 'Ordinal Certainty Equivalent' Hypothesis," *Econometrica*, pp.1045-1060.

__________(1979), "An OCE Analysis of the Effect of Uncertainty on Saving under Risk Preference Independence," *Review of Economic Studies*, 46, pp. 73-82.

[14] Thaller, R., (1980), "Towards a Positive Theory of Consumer Choice", *Journal of Economic Behavior and Organization*, 1, pp. 39-60.

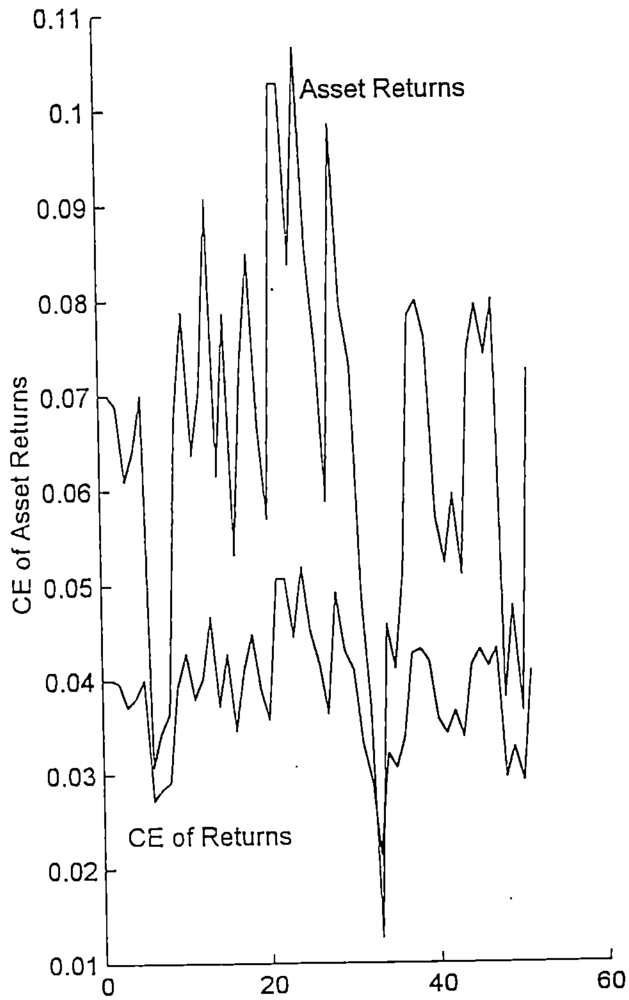


Figure 1

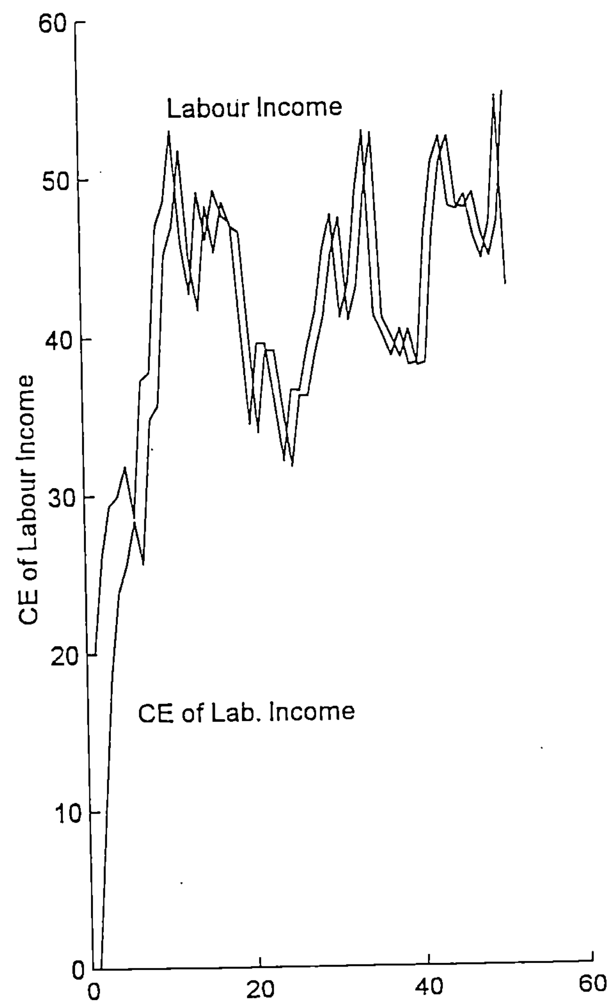
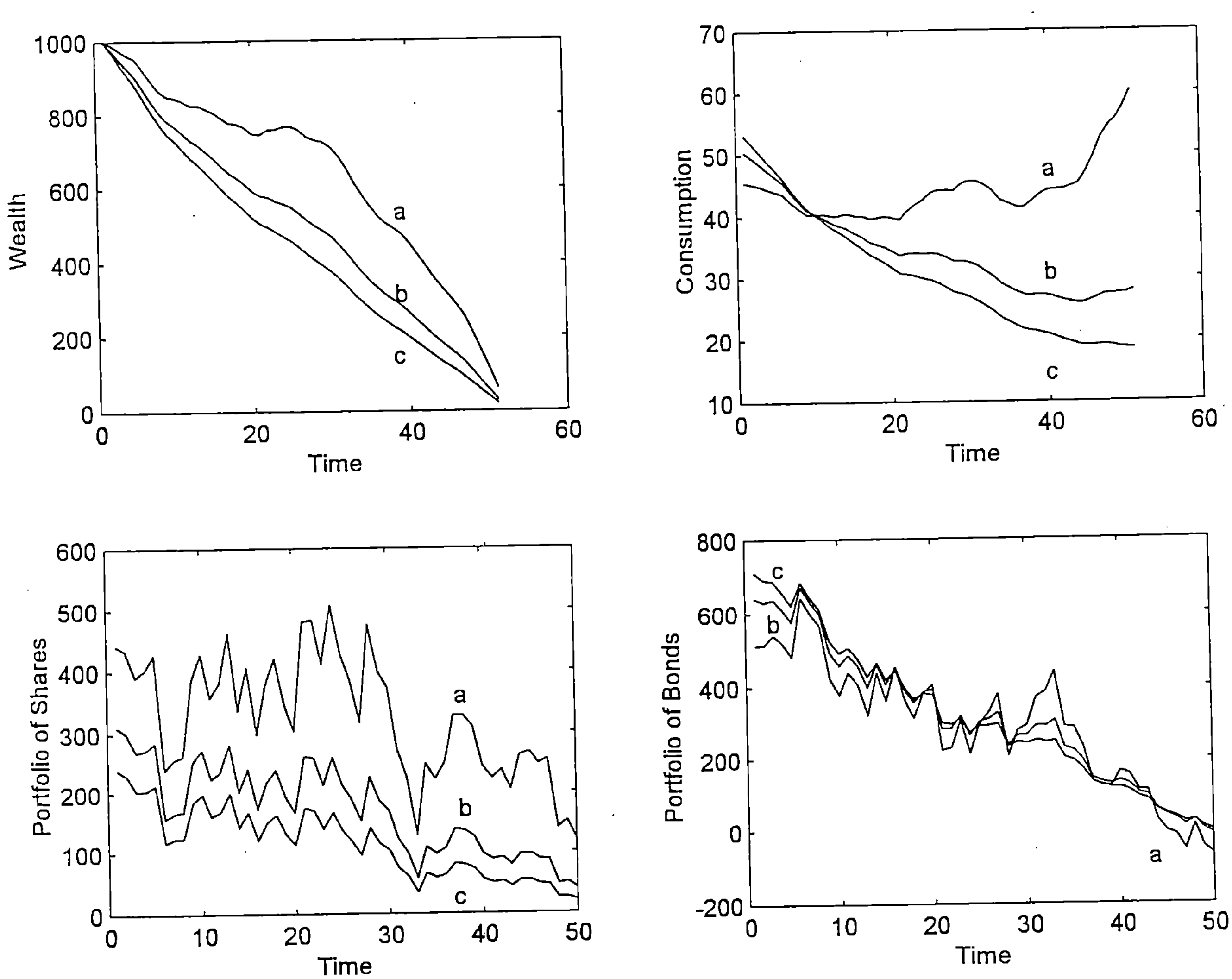
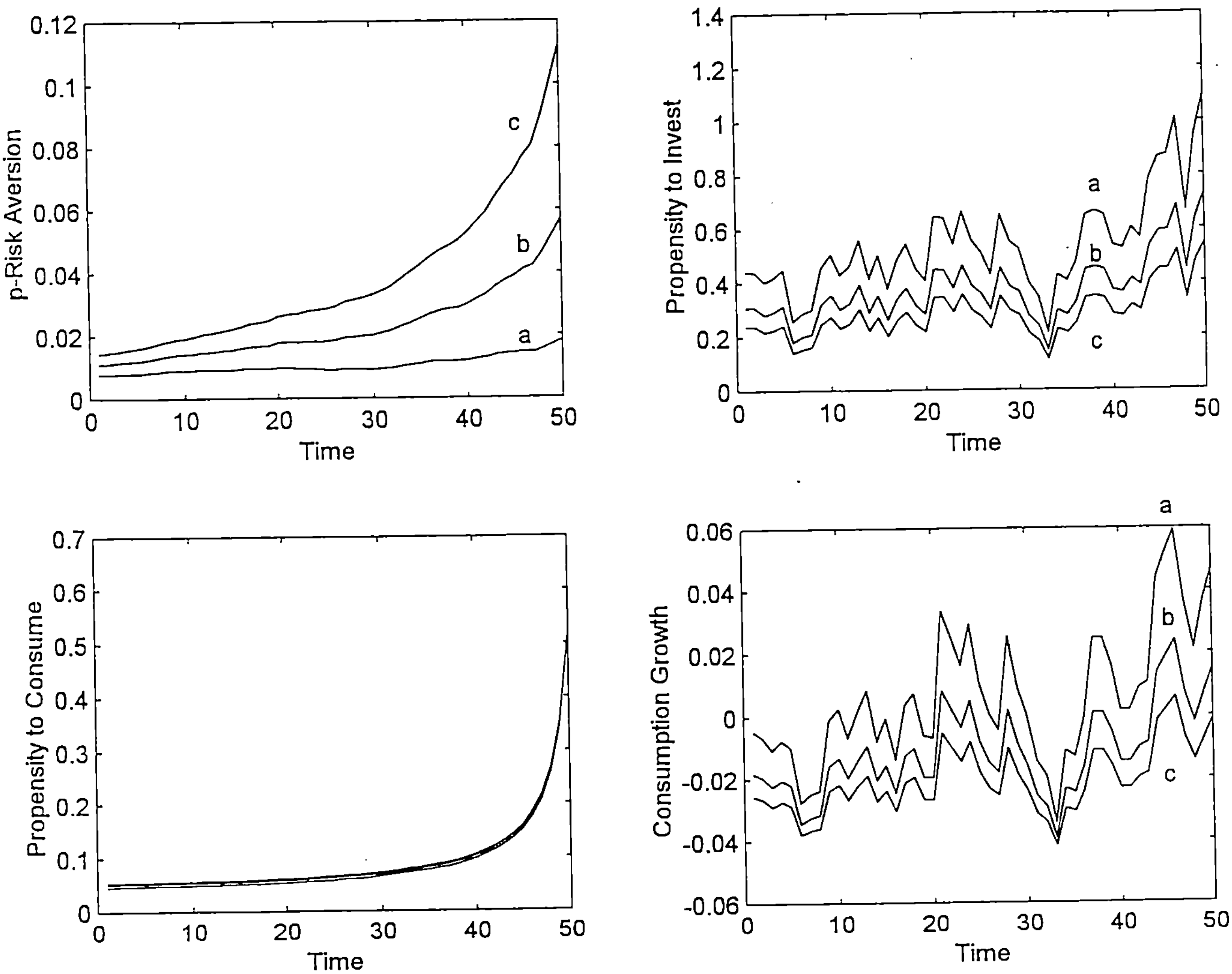


Figure 2

Simulation 1

Initial Wealth: A0 = 1000.
Zero Labour Income for all t
Parameters: $\beta = 0.975$, $\rho = 0.7$
Lines: a $\Longrightarrow \lambda = 2$
b $\Longrightarrow \lambda = 3$
c $\Longrightarrow \lambda = 4$





p-Risk Aversion
Time
c
b
a
Propensity to Invest
Time
a
b
c
Propensity to Consume
Time
Consumption Growth
Time
a
b
c

Simulation 2

Initial Wealth: A0 = 0

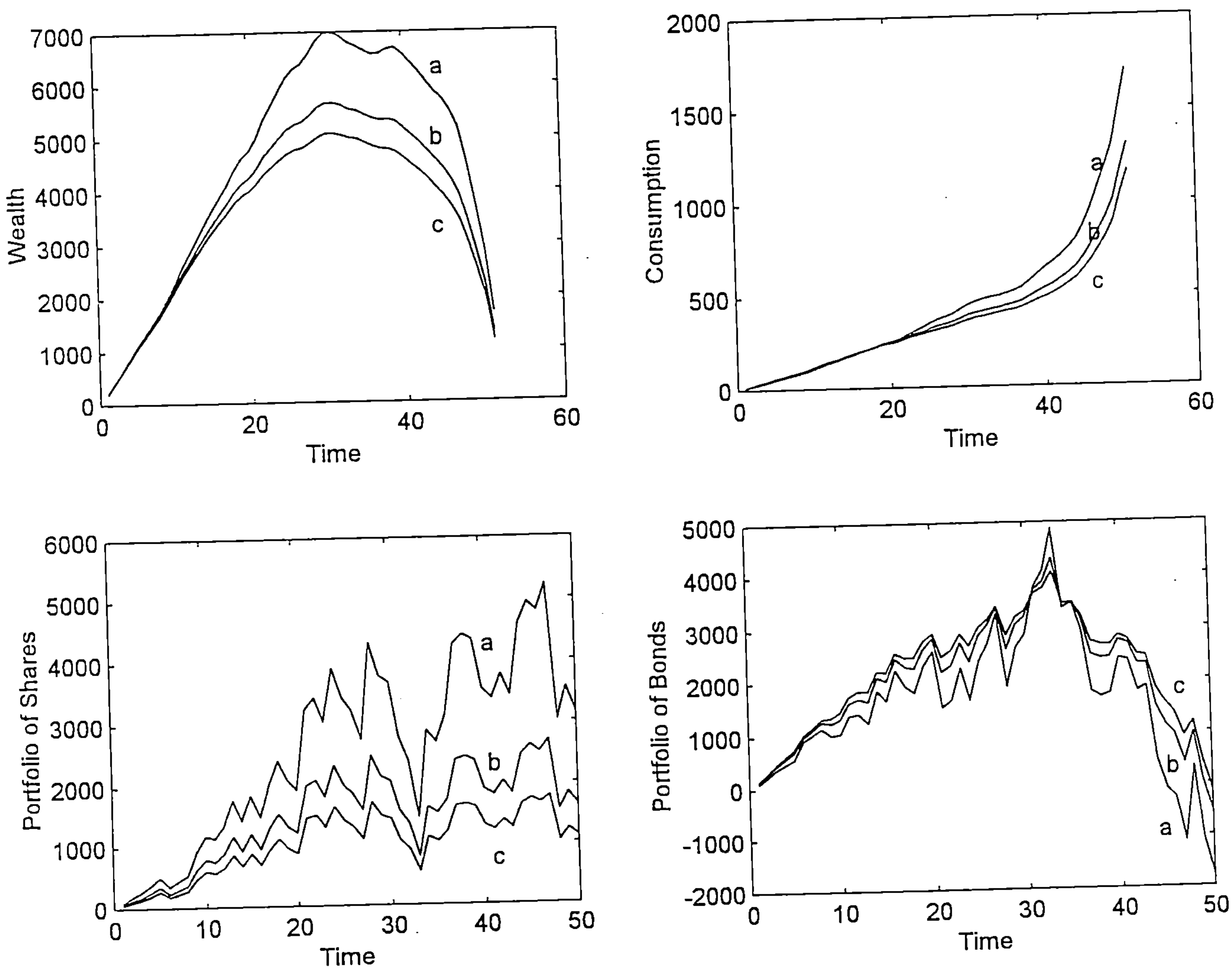
Initial Labour Income = 100

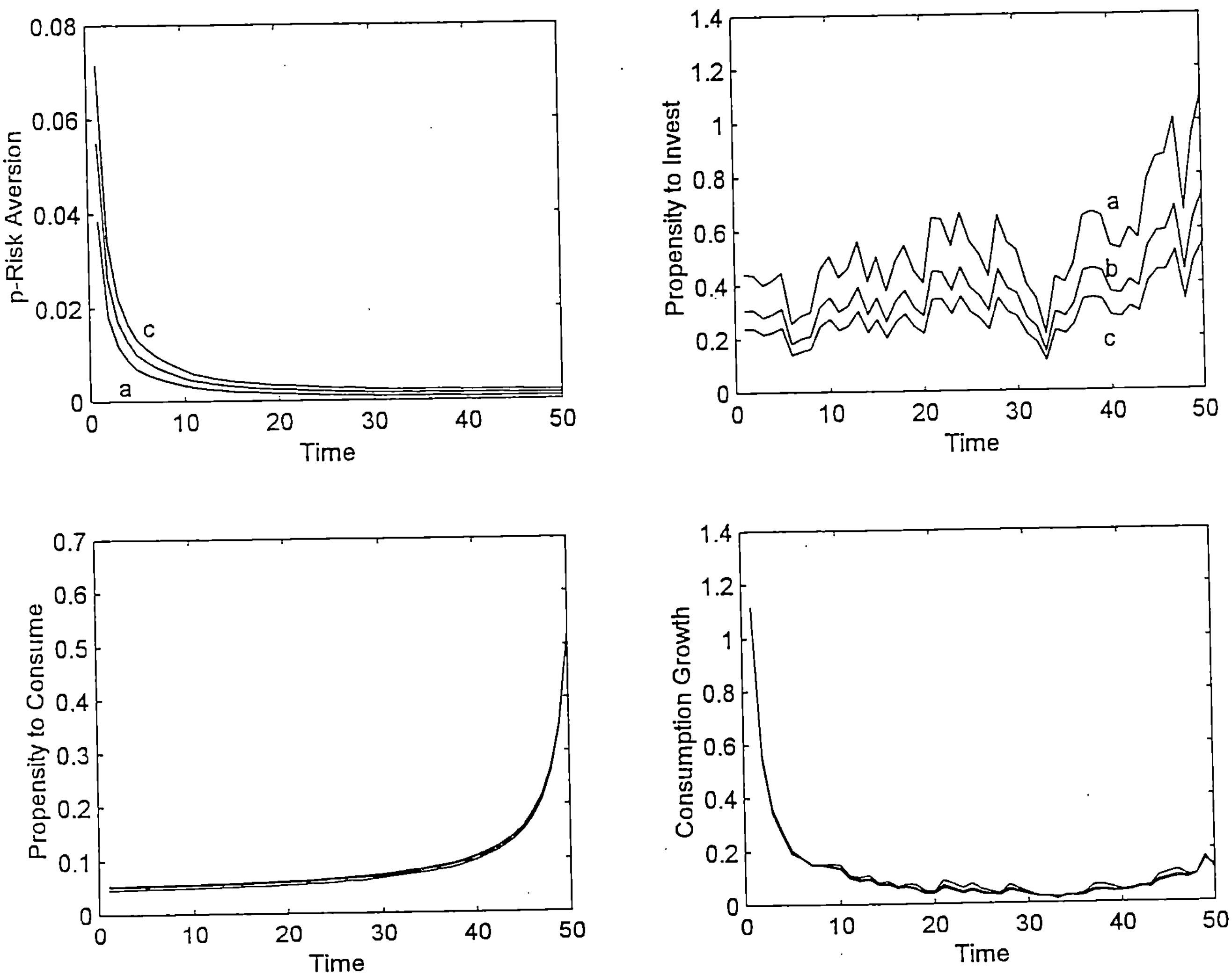
Parameters: $\beta = 0.975$, $\rho = 0.7$

Lines: a $\Longrightarrow \lambda = 2$

b $\Longrightarrow \lambda = 3$

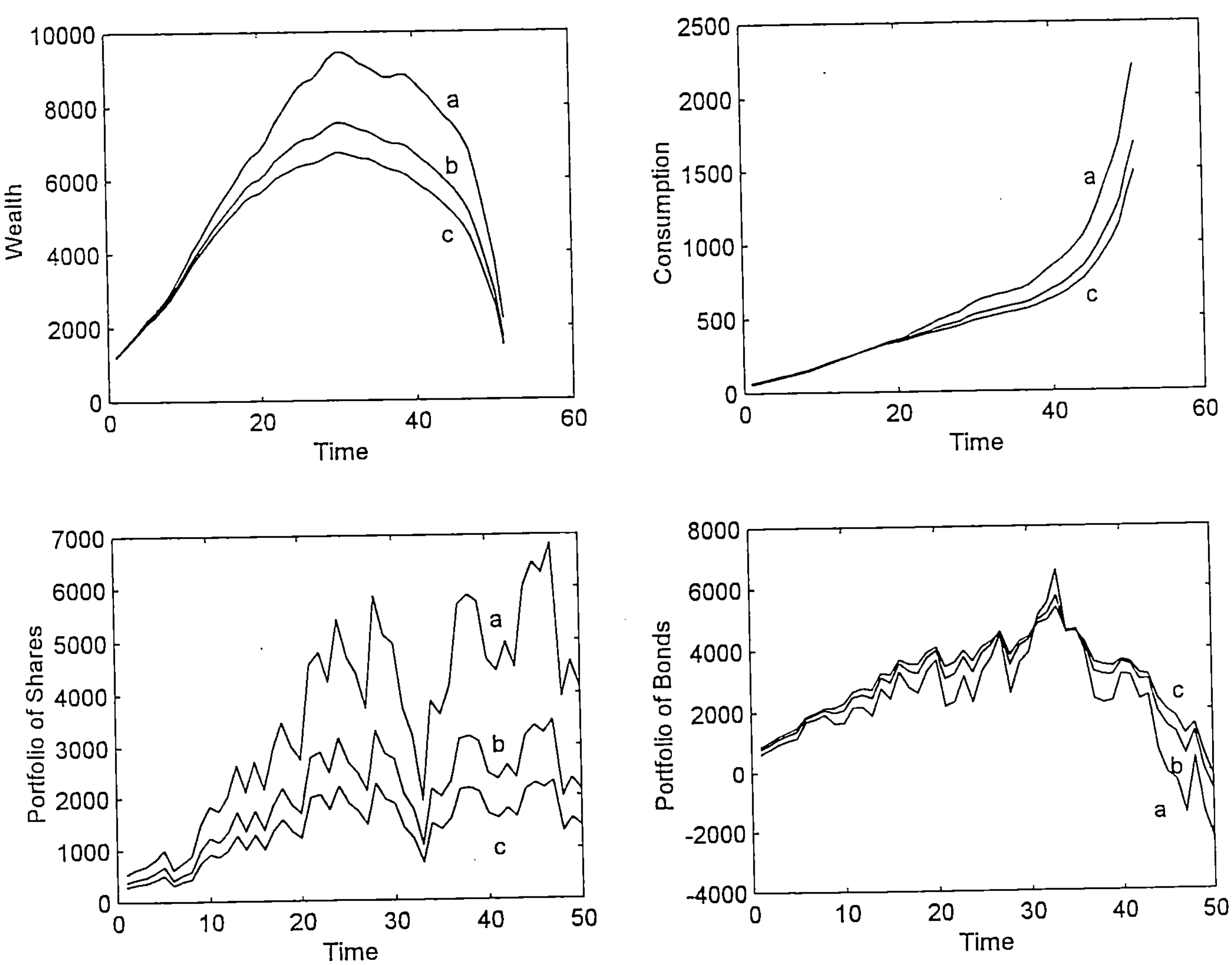
c $\Longrightarrow \lambda = 4$

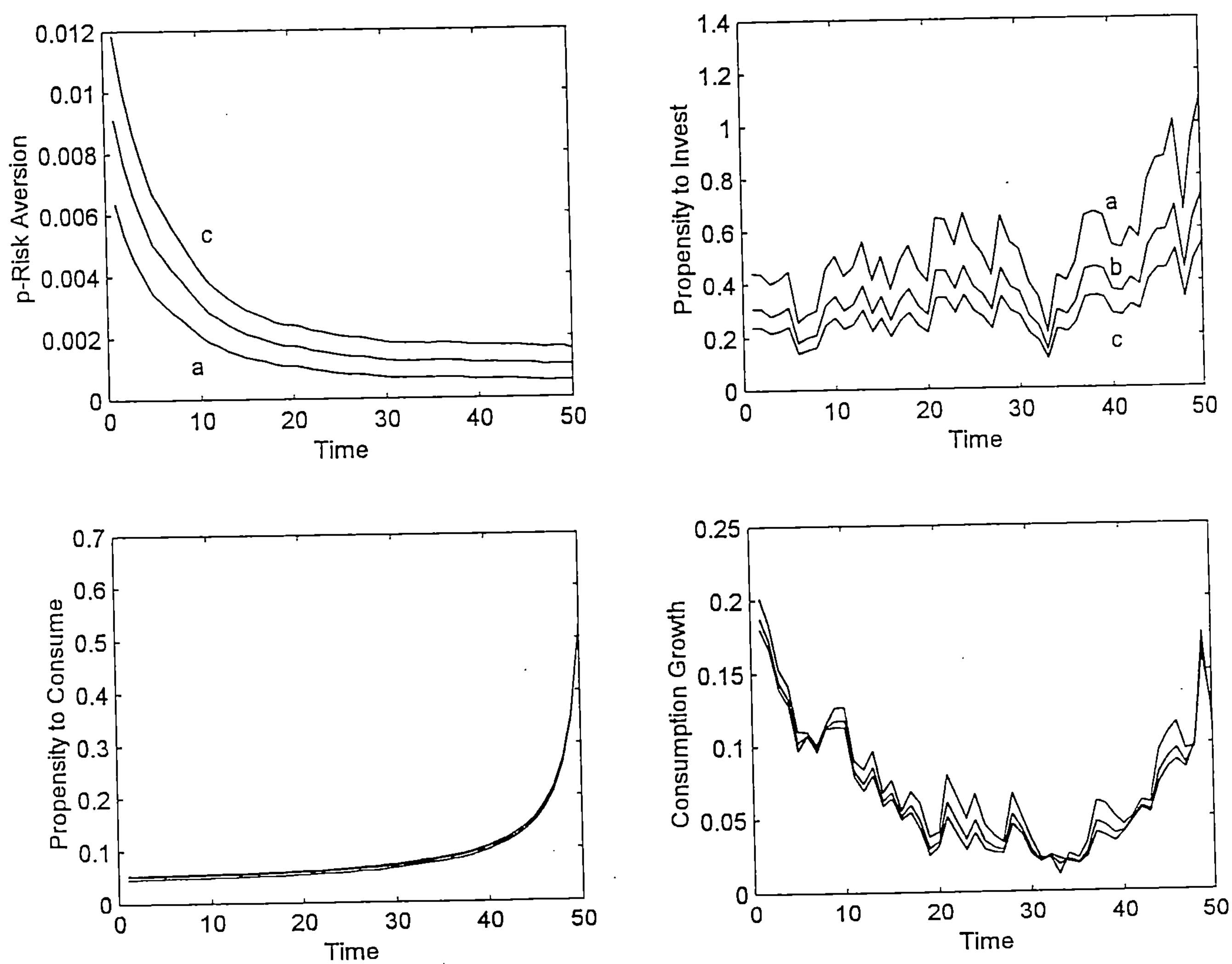



p-Risk Aversion
Time
a
c
Propensity to Invest
Time
a
b
c
Propensity to Consume
Time
Consumption Growth
Time

Simulation 3

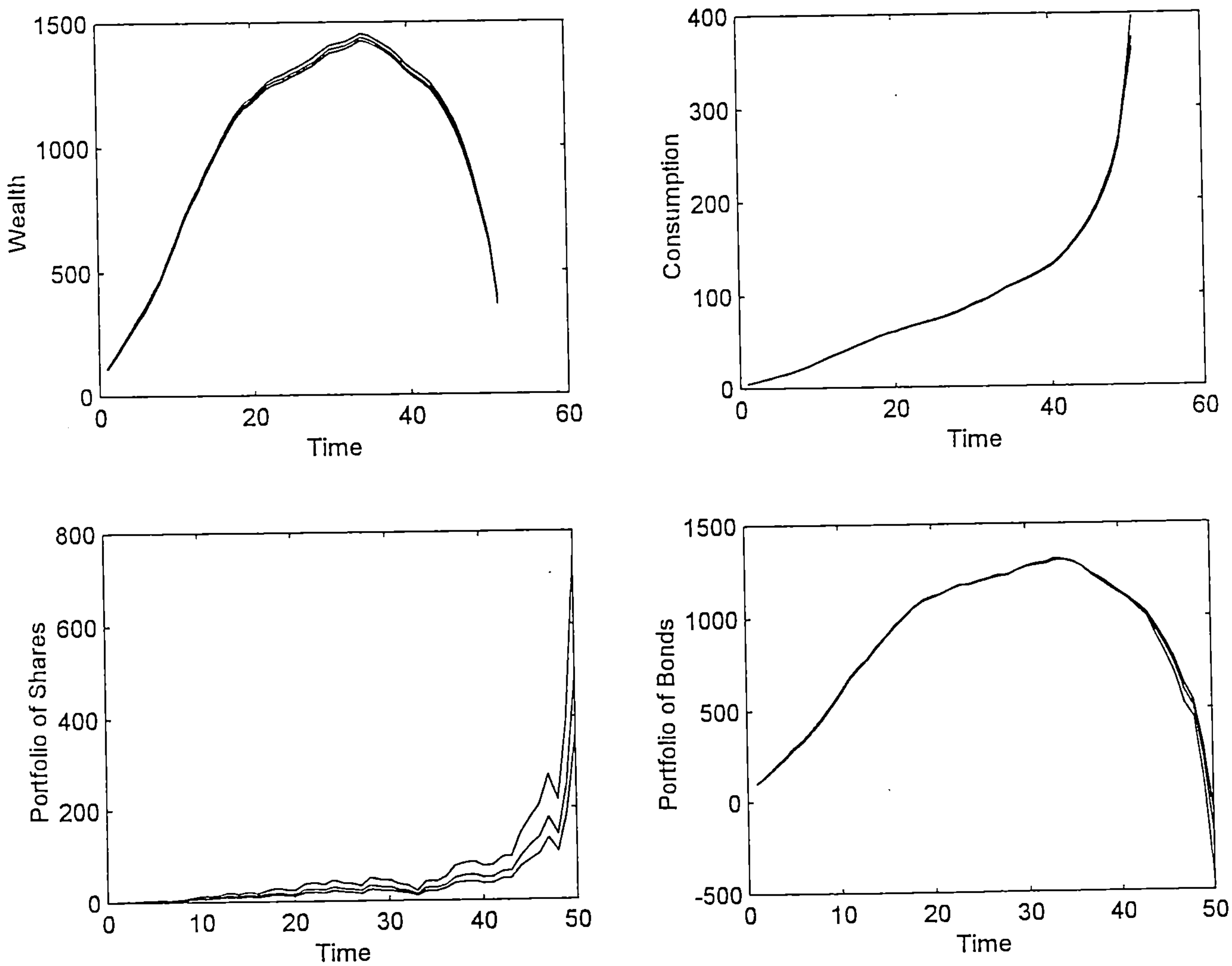
Initial Wealth: A0 = 1000.
Initial Labour Income =100
Parameters: $\beta = 0.975$, $\rho = 0.7$
Lines: a $\Longrightarrow \lambda = 2$
b $\Longrightarrow \lambda = 3$
c $\Longrightarrow \lambda = 4$

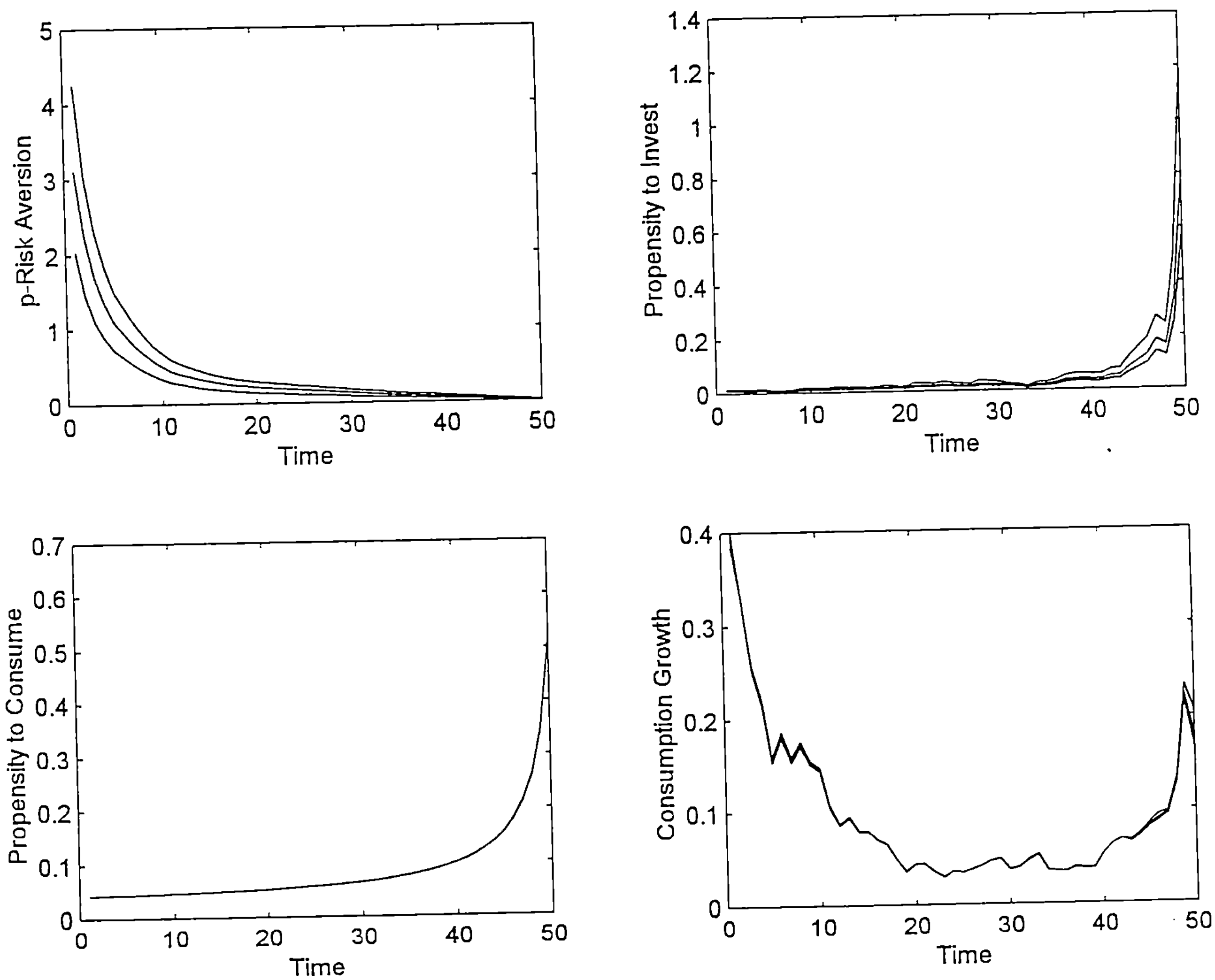



p-Risk Aversion
0.012
0.01
0.008
0.006
0.004
0.002
0
10
20
30
40
50
Time
a
b
c
Propensity to Invest
1.4
1.2
1
0.8
0.6
0.4
0.2
Propensity to Consume
0.7
0.6
0.5
0.3
0.1
Consumption Growth
0.25
0.15
0.05

Simulation 4

Initial Wealth: A0 = 1000.
Initial Labour Income =100
Parameters: $\beta = 0.975$, $\rho = 0.4$
$\lambda = \{2, 3, 4\}$

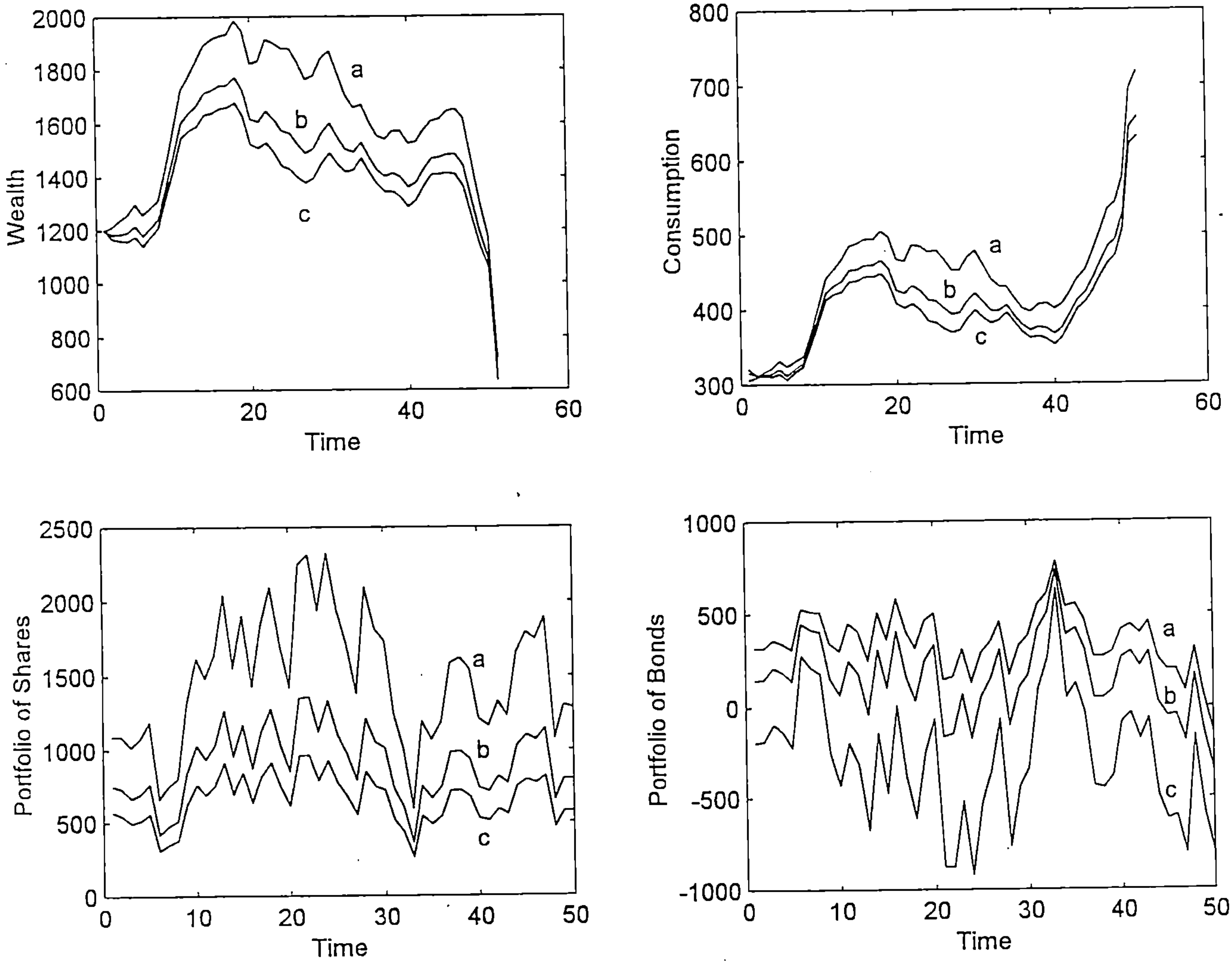


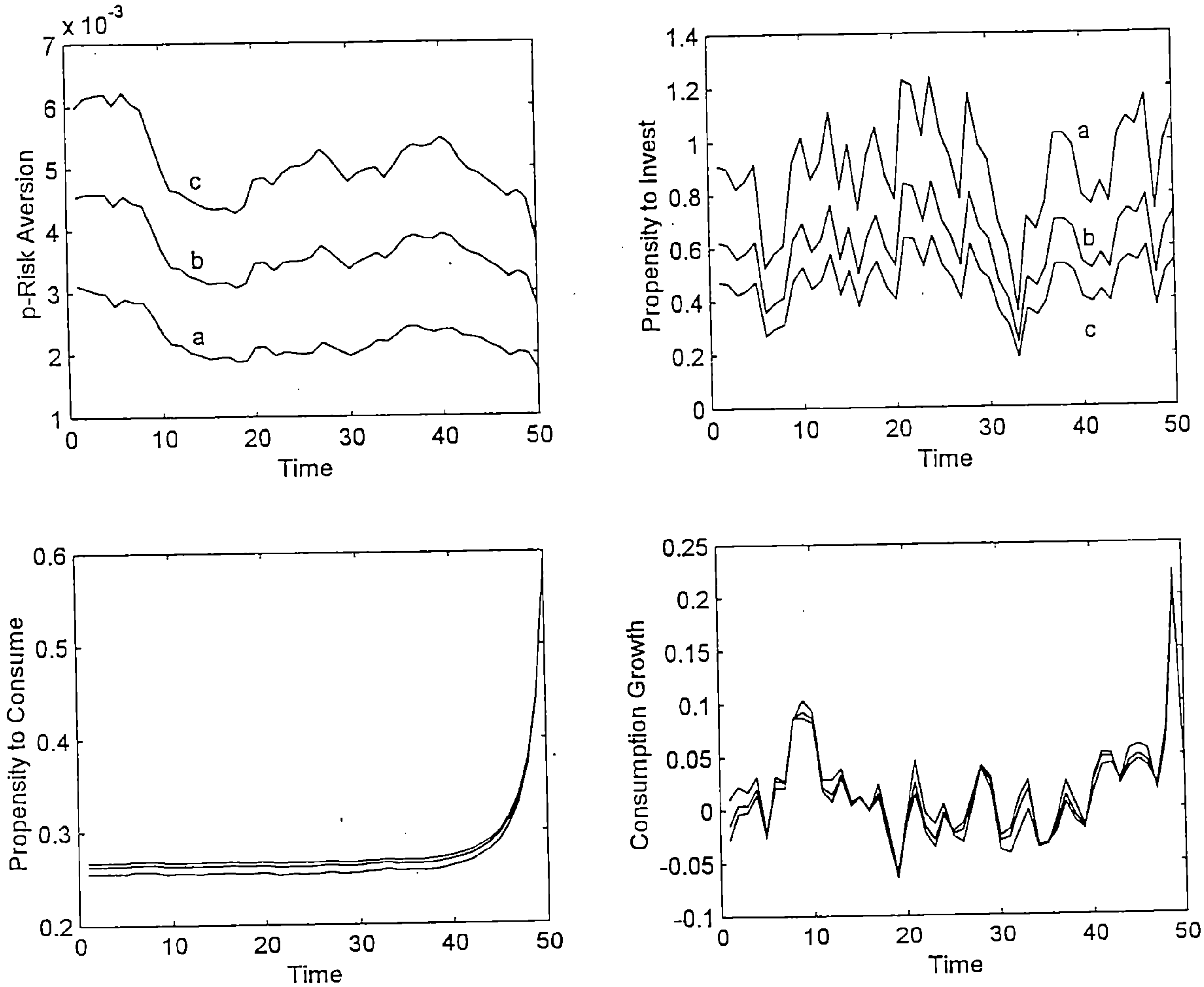


p-Risk Aversion
Time
Propensity to Invest
Time
Propensity to Consume
Time
Consumption Growth
Time

Simulation 5

Initial Wealth: A0 = 1000.
Initial Labour Income =100
Parameters: $\beta = 0.9$, $\rho = 0.7$
Lines: a $\Longrightarrow \lambda = 2$
b $\Longrightarrow \lambda = 3$
c $\Longrightarrow \lambda = 4$




x 10^-3
p-Risk Aversion
Time
a
b
c
Propensity to Invest
Time
a
b
c
Propensity to Consume
Time
Consumption Growth
Time

Simulation 6

Initial Wealth: A0 = 1000.

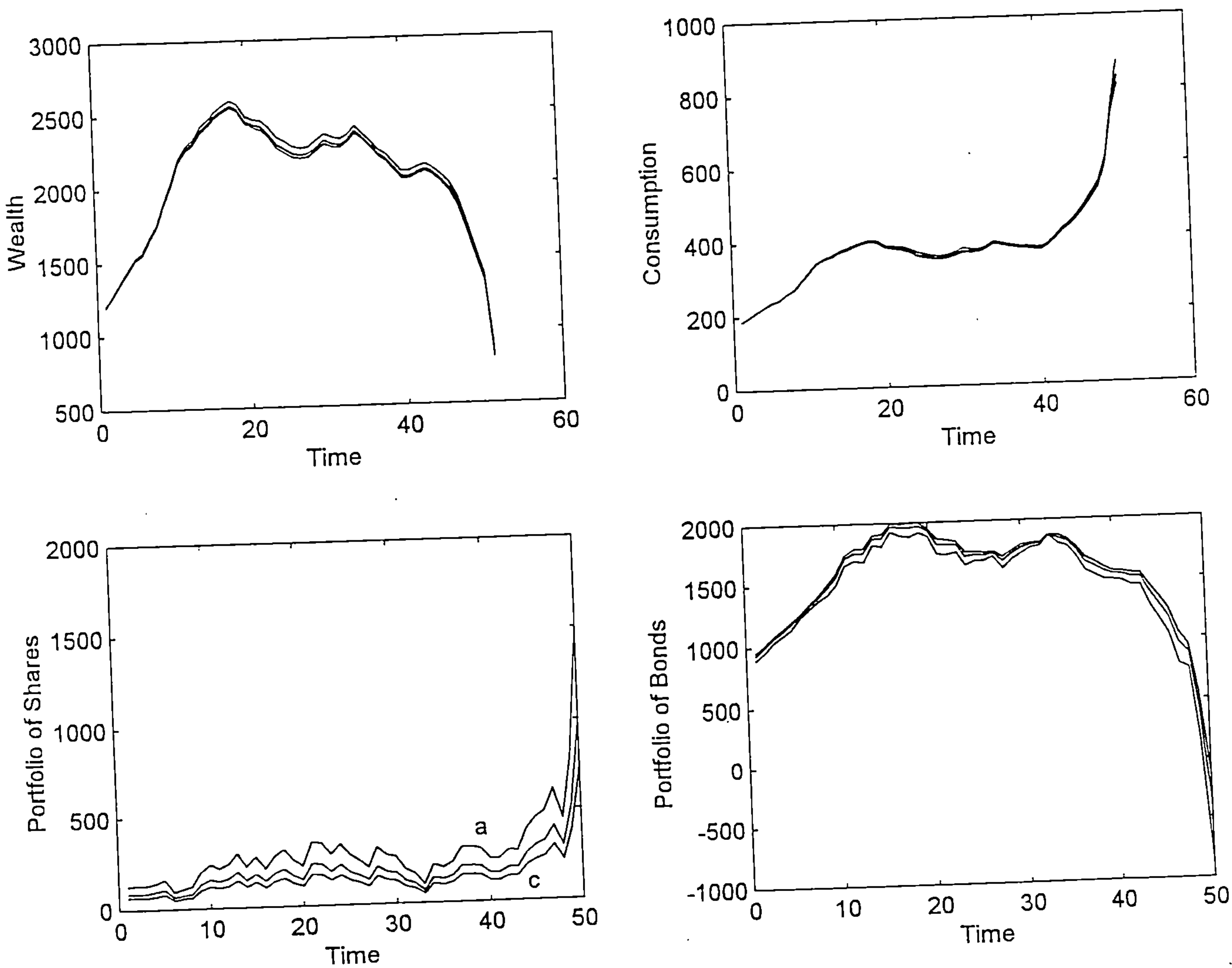
Initial Labour Income =100

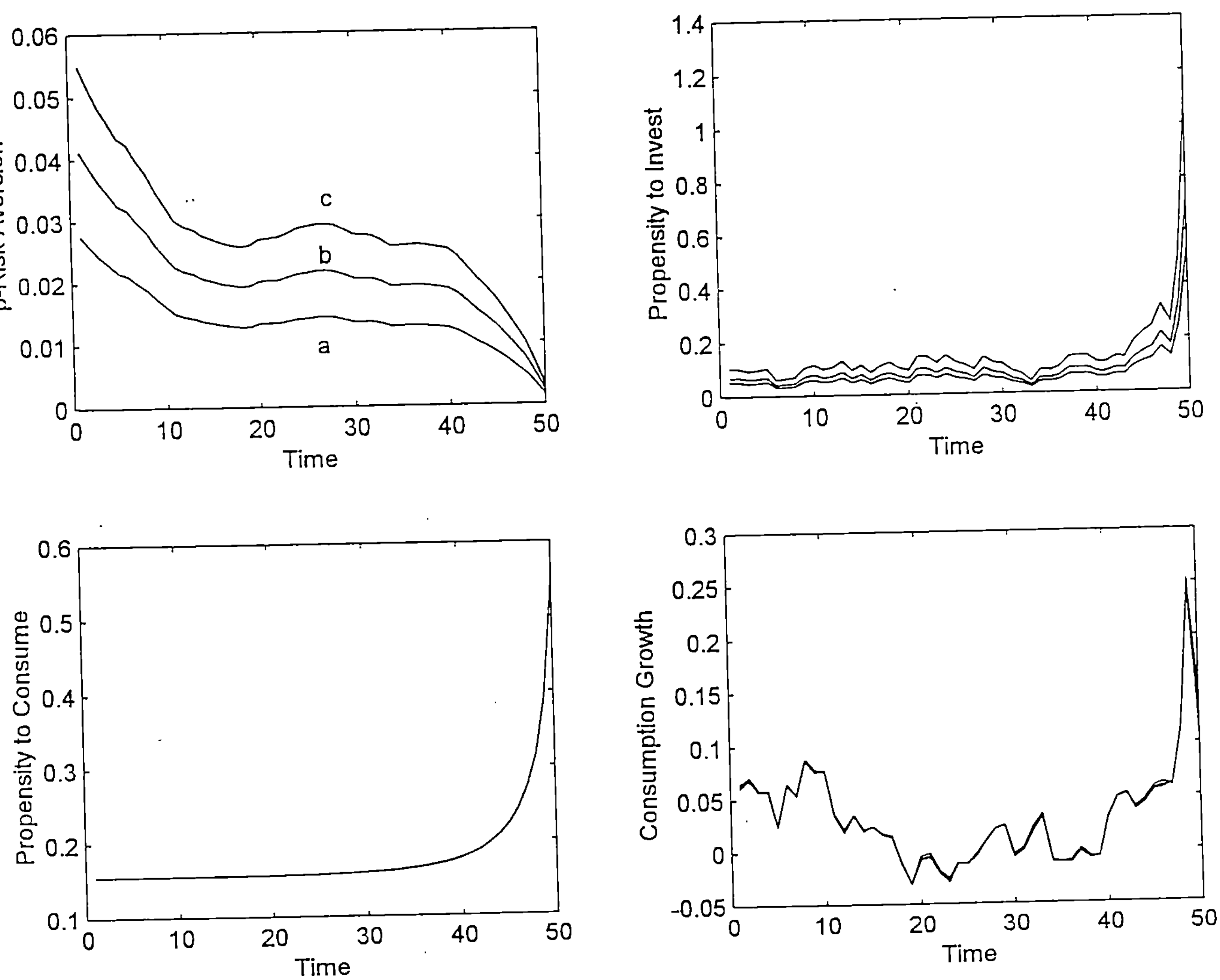
Parameters: $\beta = 0.9$, $\rho = 0.4$

Lines: a $\Longrightarrow \lambda = 2$

b $\Longrightarrow \lambda = 3$

c $\Longrightarrow \lambda = 4$




p-Risk Aversion
a
b
c
Time
Propensity to Invest
Time
Propensity to Consume
Time
Consumption Growth
Time